CAF-123 Seat No.

B. Sc. (Sem. V) Examination

December – 2021

Mathematics : Paper - CCMAT-501

(Group Theory)

Time : 3 Hours] [Total Marks : 70

Instructions : (i) All questions are compulsory.
(ii) Numbers to the right indicates the marks of each question.

1 (a) Show that the set of four transformations f_1, f_2, f_3, f_4 defined on the set of complex numbers as, $f_1(z) = z, f_2(z) = -z, f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$ forms a finite abelian group of order four w.r. to compositions of functions. 6

OR

(a) If H is a subgroup of a finite group G, then prove that $O(H)/O(G)$ 6

(b) Attempt any Two: 12

(1) Define order of an element of a group G. If a any element of group G, then show that

(i) $O(a^k) \le O(a)$, for some integer k

(ii) If $O(a) = n$ and p is prime to n, then $O(a^p) = n$.

(2) If G = (Z, +) and H = nZ, then obtain all right cosets of H in G and find the index of H in G.

(3) State and prove the Euler's theorem. If P is an odd prime then show that $1^{P-1} + 2^{P-1} + \dots + (P-1)^{P-1} \equiv (-1) \mod P$

2 (a) Define Normal subgroup. If a cyclic subgroup H of a group G is normal in G, then show that any subgroup of G is also normal in G. 6

OR

(a) Define the term 'Permutation'. Prove that out of $n!$ permutations on n symbols, $\frac{n!}{2}$ are even and $\frac{n!}{2}$ are odd permutations. 6

(b) Attempt any Two: 12

(1) Define Transposition and Disjoint cycles. Prove that any two disjoint cycles in S_n are commutative. https://www.hnguonline.com

(2) Let N be a subgroup of group G such that $a^2 \in N, \forall a \in G$, then show that N is normal subgroup of G.

(3) Let $G = C - \{0\}$ and

$$G' = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} / a, b \in R, a^2 + b^2 \neq 0 \right\}$$

be two groups under multiplication. Define $f : G \to G'$ by $f(Z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, for $Z = a + ib \in C - \{0\}$ then show that f is an isomorphism.

3 **(a)** Let $G = \langle a \rangle$ is a finite cyclic group with $O(G) = n$. Prove that for every positive divisor m of n, $H = \left\langle a^{\frac{n}{m}} \right\rangle$ is a unique subgroup of G of order m. **7**

OR

(a) Let $G = (Z, +)$ and $G' = (Z_n, +_n)$ be two groups. Define a mapping $\phi : G \to G'$ as $\phi(k) = [k], k \in G$, then show that ϕ is an onto homomorphism with kernel $K_\phi = nZ$. Deduce that $Z/nZ \cong (Z_n, +_n)$. **6**

(b) Attempt any Two: **12**

(1) If $G \neq \{e\}$ is a group having no proper subgroup, then show that G is a cyclic group of prime order.

(2) Prove that every subgroup of a cyclic group is cyclic.

(3) Show that if G is a cyclic group of prime order, then a homomorphism $\phi : G \to G'$ is either an isomorphism or $\phi(a) = e$, for each $a \in G$.

4 Attempt any Four **16**

(1) If a is any element of group G is the only element of order 2 then, show that $a \in G$ commutes with every $x \in G$.

(2) Solve $7^{1225} \equiv x \pmod{10}$

(3) Show that $f \in S_{15}$,

where $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & 10 & 13 & 4 & 5 & 12 & 3 & 15 & 2 & 9 & 11 & 6 & 1 & 14 & 8 \end{pmatrix}$

is an odd permutation. Find $O(f)$

(4) Show that every isomorphic image of a cyclic group is cyclic.

(5) Give an example of a finite abelian group of order four which is not cyclic.

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