**KA-473** Seat No. ________

**B. Sc. (Sem. V) Examination**

**October / November – 2017**

**Mathematics : CC-MATH-501**

*(Group Theory)*

Time : 3 Hours] [Total Marks : 70

**Instructions :** (1) All questions are **compulsory**.

(2) The **figures** to the **right** indicate the marks of the corresponding questions.



1 (a) Define a group. 6

Examine each set given below and determine whether it is a group under the binary operation $*$. If it is a group, then obtain its identity and if it is not a group then find out which postulates are not satisfied.

(i) Set $\mathbb{Z}$ with $a*b=a-b$

(ii) Set $\mathbb{N}$ with $a*b=a\cdot b$

(iii) Set $\{z\in\mathbb{C} \,/\, |z|=1\}$ with $a*b=a\cdot b$

(b) State and prove Lagrange's theorem. 6

(c) In a finite group, prove that each element is of a finite order. 6

**OR**

1 (a) State and prove the necessary and sufficient conditions for a non-empty subset H of a group G to be a subgroup of G. 6







(b) Prove that a group G is commutative if $(ab)^n=a^nb^n$, $a, b\in G$, for three consecutive integer n. 6

(c) If $G=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \middle| a,b,c,d\in\mathbb{Z}_2\right\}$. 6

Show that G is a commutative group under matrix addition. Also find the order of G.



2 (a) Prove that the order of a permutation $f\in S_n$ is the least common multiple of the length of its disjoint cycles. 6

(b) If $G=\left\{e,a,a^2,a^3,\ldots\ldots,a^{19} \middle| a^{20}=e\right\}$ is a cyclic group of order 20 and $H=\left\langle a^4\right\rangle$, then prepare the group table for the quotient group G/H Using group table, answer the following 6

(i) Find the inverse of $Ha^3$ in G/H

(ii) Solve the equation $\left(Ha^3\right)x=Ha^2$ in G/H

(c) Show that isomorphism between two groups is an equivalence relation. 6

**OR**

2 (a) Show that any two disjoint cycles in $S_n$ are commutative. 6

(b) If for a subgroup H of a group G, the product of two right cosets of H in G is again a right coset of H in G then prove that H is a normal subgroup of G. 6

(c) In usual notations, if $G$ is a group with $a(G) = \{I_G\}$ then show that 6

(i) $G$ is commutative

(ii) $a^2 = e$ for each $a \in G$.

3 (a) Show that, in a finite cyclic group, the group and its generator have the same order. 6

(b) Show that a cyclic group of order $n$ has exactly $\phi(n)$ generators. Where $\phi$ is the Euler's phi function. 6

Moreover, what can you say about the generators of an infinite cyclic group ?

Justify your answer.

(c) Show that Kernel $K_\phi$ of a homomorphism $\phi : (G, 0) \to (G', *)$ is a normal subgroup of $G$. 6

**OR**

3 (a) State and prove the first fundamental theorem of homomorphism. 6

(b) Show that a cyclic group of order eight is homomorphic to a cyclic group of order four. 6

(c) If $G \neq \{e\}$ is a group having no proper subgroup then show that $G$ is a cyclic group of prime order. 6











4 Attempt any **two** : 8

(a) Show that a group of order five is always commutative.

(b) Show that the set

$H = \{ f \in S_n \mid 1 \text{ is invariant under } f \}$ is a subgroup of $S_n$.

(c) The set $G = R \sim \{-1\}$ is a group under binary operation $*$, where $a * b = a + b + ab$, $a, b \in G$ and if $G' = (R_0, \bullet)$, where $R_0$ is the set of all non-zero real numbers, then show that $G \cong G'$.

5 Attempt any **two** : 8

(a) Using the Euler's theorem, find the remainder obtained on dividing $3^{256}$ by 14.

(b) Prove that

(i) A subgroup of index 2 in a group is a normal subgroup.

(ii) The alternating subgroup $A_n$ of symmetric group $S_n$ is a normal subgroup of $S_n$ for each $n \geq 2$.

(c) Prove that a homomorphism $\phi : (G, 0) \to (G', *)$ is one-one if and only if $K_\phi = \{e\}$.